

A Casino Game

According to legend, the young Gauss responded in a matter of moments to his teacher's disciplinary assignment that he must add the integers $1 + 2 + 3 + \dots + 100$. Gauss saw the possibility of stacking two versions of the same sum, the second in reverse order, like so:

$$\begin{array}{cccccccc}
 1 & + & 2 & + & 3 & + & \dots & + & 99 & + & 100 \\
 100 & + & 99 & + & 98 & + & \dots & + & 2 & + & 1 \\
 \hline
 101 & & 101 & & 101 & & \dots & & 101 & & 101.
 \end{array}$$

He then added the column sums, 100×101 , and divided the result by 2 to get the correct answer. A generalization of this idea lies behind a first approach to an optimal placement of the dice in our Casino Game.

Exercise 1. Suppose two rearrangements (permutations) of the integers $0, 1, 2, \dots, n-1$ are lined up one above the other with the result that all the column sums equal the same number x . What is x ?

Exercise 2. Suppose three rearrangements of the integers $0, 1, 2, 3, 4, 5, 6$ are lined up one above the other with the result that all the column sums equal the same number x . What is x ? If it is possible to do this for the integers $0, 1, 2, \dots, n-1$, what restriction on n is implied? What is x in the general case?

Before continuing, let's get a handle on the probability (or outcome) space for the Casino Game. Given any placement of the dice, there are 49 different stopping positions of the inner concentric rings with respect to the outer ring. Each stopping position yields 7 radial sums, i.e., 7 sums of lined-up dice. Thus, once the dice are placed, forty-nine 7-tuples of radial sums constitute the set of possible outcomes. Some of these 7-tuples may contain at least one 7 and one 11 as radial sums. We'll use *Winners* to denote the number of such "7&11" outcomes for any given placement of the dice. Some of the 49 outcomes may contain seven 9's as radial sums. *SuperJackpots* will be used to denote the number of these outcomes.

It is entirely possible (*Exercise 3*, below) that an unlucky placement of the dice will have $Winners = SuperJackpots = 0$, resulting in an expected value of $-\$1.00$ per game. In general, the expected value E of the game, in dollars, is

$$E = \frac{Winners}{49}(2) + \frac{Superjackpots}{49}(50) - 1.$$

Exercise 3. Verify that the placement $[0\ 1\ 2\ 3\ 4\ 5\ 6]$, $[0\ 1\ 2\ 3\ 4\ 5\ 6]$, $[0\ 6\ 4\ 2\ 5\ 3\ 1]$ has $E = -1$, and find a *different* arrangement that also has $E = -1$. *Hint:* replace all dice values x with $6 - x$. Why does this maneuver leave the expected value of any placement unchanged?

It is important to indicate what we mean by *different* in *Exercise 3*, above. We do *not* mean an arrangement that simply uses the same vectors playing different roles as the outer, middle, and innermost rings,

respectively. We do not mean the three vectors written in reverse order, or those obtained by applying shifts-with-wraparound (*cyclic shifts*). On the contrary, these would be considered the *same* arrangement due to the essential sameness of the outcome space.

Heading in the desired direction, if we could find an arrangement that had just one SuperJackpot (one out of forty-nine outcomes with seven radial sums equal to 9, and no “7&11” winners) E would be about two cents – not much, but still *positive*. Can we do better? The answer is: Much better!

Generalizing *Exercise 2*, let the top, middle, and bottom rows of the $3 \times n$ matrix, below, represent the top, middle, and innermost rings of three rotating disks with an extended range of “dice values”. Let

$n \geq 3$ be odd and $p = \frac{n-1}{2}$.

$$\begin{bmatrix} 0 & 1 & 2 & \cdots & p-1 & p & p+1 & p+2 & \cdots & n-2 & n-1 \\ p & p+1 & p+2 & \cdots & n-2 & n-1 & 0 & 1 & \cdots & p-2 & p-1 \\ n-1 & n-3 & n-5 & \cdots & 2 & 0 & n-2 & n-4 & \cdots & 3 & 1 \end{bmatrix}$$

Note that all the column sums are equal and that the first two rows are the same (up to a cyclic shift). When $n = 7$, each column sum is 9. That represents a SuperJackpot. However, there’s something more surprising about this arrangement with $n = 7$: There are *two* outcomes out of 49 with 7 radial sums equal to 9, giving an expected value of about \$1.04! Can we do better? The answer is: See *Exercise 6*, below.

Exercise 4. Why do two equivalent rows (modulo a cyclic shift) in the $3 \times n$ matrix, above, imply two stopping positions with equal column sums?

Here’s a different arrangement with equal column sums and two equivalent rows, hence two stopping positions with equal column sums:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 \\ p & n-1 & p-1 & n-2 & p-2 & \cdots & 2 & p+2 & 1 & p+1 & 0 \\ n-1 & p-1 & n-2 & p-2 & n-3 & \cdots & p+2 & 1 & p+1 & 0 & p \end{bmatrix}.$$

Exercise 5. Prove that the greatest possible number of “7&11” winners is 33. However, we conjecture that 27 is the true maximum number of “7&11” winners. Can you find an arrangement that achieves this number? *Hint:* Start with [0 1 2 3 4 5 6] and [0 6 5 4 3 2 1].

Exercise 6. Compare the performance in the Casino Game of the following two arrangements:

$$A_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 0 & 2 & 4 & 1 \\ 4 & 5 & 1 & 6 & 3 & 0 & 2 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & 0 & 3 & 2 & 5 & 4 & 6 \\ 3 & 5 & 0 & 6 & 4 & 2 & 1 \\ 5 & 4 & 6 & 1 & 0 & 3 & 2 \end{bmatrix}.$$

What simple operation has been applied to A_1 in order to obtain A_2 ? This operation does something interesting to two of the rows of A_1 .

Exercise 7. Use the hint in *Exercise 3*, above, to find a second matrix like A_1 in *Exercise 6*, i.e., one that has as its first row $[0\ 1\ 2\ 3\ 4\ 5\ 6]$ and all column sums equal to 9.

Exercise 8. Find all 13 different matrices, of which A_1 in *Exercise 6* is one, that have $[0\ 1\ 2\ 3\ 4\ 5\ 6]$ as the first row and each column sum equal to 9.

Exercise 9. Find the probability that a random arrangement of the dice has *at least* one SuperJackpot.

Open problems: programming project. What is the maximum expected value of this game? What are all the arrangements that have this value? Is the conjecture of *Exercise 5* correct? How many different arrangements have the maximum number of “7&11” winners?

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