Mathematics of an Ancient Computing Device

Suppose that the integer values w_j of the weights of a two-pan balance scale are written in nondecreasing order as $w_1 \le w_2 \le \cdots \le w_N$. Let $S_k = \sum_{j=1}^{k} w_j$, where $S_0 \equiv 0$. The necessary and sufficient condition for exact measurement of any integer weight X in the range $0 \le X \le S_N$ is

$$S_{k+1} \le 3S_k + 1, \quad k = 0,1,\cdots,N-1,$$

or equivalently, (1)

$$w_{k+1} \le 2S_k + 1, \quad k = 0,1,\cdots,N-1.$$

Proof. To demonstrate the sufficiency, i.e., that the satisfaction of (1) implies every integer weight X in the range $0 \le X \le S_N$ can be weighed exactly, we proceed by induction. Assuming that the scale is not defective, it balances when nothing at all is placed on it. So the trivial weight $X = 0$ can be weighed exactly.

For $1 \le k \le N$, the induction hypothesis asserts that all integer weights X in the range $0 \le X \le S_{k-1}$ are weighable. If so, we want to show that all X in the range

$$S_{k-1} < X \le S_k \tag{2}$$

are weighable. However, it follows from (2) and (1) that

$$|X - w_k| \le S_{k-1}. \tag{3}$$

Inequality (3) will be verified momentarily, but the important point is that $X' = |X - w_k|$ is a new unknown weight that falls within the induction range: Think of X on one pan and w_k on the other as $X' = |X - w_k|$ on a single pan. So X' can be weighed exactly, from which it follows that the weight of X can be expressed in terms of the w_j's. This proves that all integer weights X in the range (2) can be weighed.

To confirm (3), suppose for a contradiction that it isn't true. The first possibility is that $X - w_k > S_{k-1}$, which implies $X > S_{k-1} + w_k = S_k$, and this violates (2). The other possibility is that $w_k - X > S_{k-1}$. In this inequality use (1) to replace w_k by $2S_{k-1} + 1 \ge w_k$ to get $2S_{k-1} + 1 - X > S_{k-1}$, which implies $S_{k-1} > X - 1$ or $X \le S_{k-1}$. Again, this violates (2).

Therefore all integer weights X in the range $0 \le X \le S_k$ can be weighed, completing the induction proof of the sufficiency of condition (1).

To prove that condition (1) is necessary if every integer weight in the range $0 \le X \le S_N$ is to be weighed, we need only find one weight that cannot be weighed if (1) is ever violated. In the chain of inequalities

$$w_1 \le \cdots \le w_{k-1} \le \underbrace{w_k \le \cdots \le w_N}_{B} \tag{4}$$

suppose that $w_k > 2S_{k-1} + 1$ is any instance where (1) is violated. The collection of weights denoted by *B* will be referred to, below. We will show that the weight $X = S_N - w_k + 1$ can never be weighed by actually trying to do it and seeing what happens.

Place *X* on the left-hand pan of the scale. No weight w_m from *B* may be placed along with *X* because $X + w_m = S_N + (w_m - w_k) + 1 > S_N$ implies that the left pan will always outweigh the right pan.

Next, suppose that *every* weight *except* one, w_m from *B*, is placed on the right-hand pan. In that case, the weight of the right pan, $S_N - w_m \le S_N - w_k < S_N - w_k + 1 = X$, is lighter than *X*. If leaving just a single weight from *B* off the right pan means that we cannot weigh *X*, it follows that we must place *every* weight from *B* on the right pan.

To keep the right pan as light as possible, suppose that we place *only* those weights from *B* on the right pan, and begin placing the remaining weights $w_1, w_2, \cdots, w_{k-1}$ along with *X* on the left pan. Now the scale will not balance because

$$X + S_{k-1} = S_N - w_k + 1 + S_{k-1} < S_N - 2S_{k-1} - 1 + 1 + S_{k-1} = \sum_{j=k}^{N} w_j = \text{the total weight of } B.$$

In other words, with just the weights from *B* on the right-hand pan and *every* remaining weight on the left-hand pan along with *X*, the right pan will remain heavier.

So if (1) is violated, we have found a weight $X = S_N - w_k + 1$ that cannot be weighed. This completes the proof that (1) is both sufficient and necessary for the exact measurement of all integer weights *X* in the range $0 \le X \le S_N$.

Exercises

1. In the basis step of the induction proof, above, a “trivial” weight *X* was shown to be measurable without using any of the w_j at all. Which *non*trivial *X* can always be weighed if (1) holds?

2. Suppose, as in (4), above, that $w_k > 2S_{k-1} + 1$ is an instance where (1) is violated. Show that the weight $S_N - 2S_{k-1} - 1$ cannot be weighed.

3. It stands to reason that the more that w_k exceeds $2S_{k-1} + 1$ the greater the range of values that cannot be weighed. Identify such values explicitly.

4. Show how (3) above leads to the simple algorithm, secreted in the applet of Part 1, for weighing an unknown weight X in the range $0 \le X \le S_N$. Suppose that an "elementary computational step" consists of placing (removing) a single weight. What is the computational complexity of this algorithm? Is there a faster method?

5. For fixed N, what choice of weights w_j maximizes the range $0 \le X \le S_N$? Show that in this case there is *unique* placement of weights that measures X.

6. Suppose that three of your original set of six balancing weights are missing, and that the remaining three weigh 1, 8, and 70 grams, respectively. If weights cost 10 cents per gram and you are willing to spend up to $20, which three weights should you purchase to maximize the range?

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